Chapter 6

THE EXPONENTIAL FUNCTION

Are you thinking of buying a computer? Moore's Law suggests that the processing power of computers doubles every eighteen months, which means that in a year and a half from today, computers will be twice as powerful as they are now! This is an example of exponential growth. In this chapter, you will study the exponential functions that can be used to describe and make predictions about the growth of biological populations, including human populations and populations of cancerous cells, the growth of financial investments, the growth of the Internet, and the decaying of radioactive substances. Another application of exponential functions occurs in psychology, where it has been noted that, in certain circumstances, there is an exponential relationship between the size of a stimulus and a nerve's response to the stimulus. The common feature in all these situations and many others is that the amount of growth or decline at any point in time is directly proportional to the size of the thing that is growing or declining.

CHAPTER EXPECTATIONS In this chapter, you will

- identify key properties of exponential functions, **Section 6.1, 6.2**
- determine intercepts and positions of the asymptotes to a graph, **Section 6.2, 6.3**
- describe graphical implications of changes in parameters, **Section 6.3**
- describe the significance of exponential growth or decay, **Section 6.4, 6.5**
- pose and solve problems related to models of exponential functions, **Section 6.4, 6.5, Career Link**
- predict future behaviour by extrapolating from a mathematical model, **Section 6.5**

Review of Prerequisite Skills

In this chapter, you will be studying the exponential function. You will require a knowledge of rational exponents, such as the following:

- $x^0 = 1$
- $x^{-n} = \frac{1}{x^n}, x \neq 0$
- $a^{\frac{p}{q}} = (\sqrt[q]{a})^p$ or $\sqrt[q]{a^p}$

In this chapter you will also be working with transformations.

Exercise

1. Find the value of each of the following:
   a. 4^3 b. $(-3)^2$ c. -3^3 d. $\left(\frac{3}{4}\right)^2$
2. Write the following with positive exponents:
   a. x^{-7} b. $5m^{-2}$ c. $(3b)^{-3}$ d. $\frac{1}{w^{-5}}$ e. $\left(\frac{2}{3}\right)^{-2}$
3. Evaluate the following:
   a. 5^{-2} b. $\left(\frac{2}{3}\right)^{-1}$ c. $\left(\frac{3}{4}\right)^{-3}$ d. $\frac{2^{-1} + 2^{-2}}{3^{-1}}$
4. Evaluate the following:
   a. $4^{\frac{1}{2}}$ b. $27^{\frac{1}{3}}$ c. $\left(\frac{1}{8}\right)^{\frac{1}{3}}$
5. a. Sketch the following functions on the same grid:
   $y_1 = x^2$, $y_2 = x^2 + 4$, and $y_3 = x^2 - 3$.
   b. Describe
      i) the transformation of the graph of y_1 to y_2.
      ii) the transformation of the graph of y_1 to y_3.
   c. Without sketching, describe the transformation of the graph of $y = x^2 - 2$ to the graph of $y = x^2 + 2$.
   d. Describe what happens when a positive or negative constant is added to a function.

6. a. Sketch the following functions on the same grid:

$y_1 = x^2$, $y_2 = \frac{1}{2}x^2$, and $y_3 = -2x^2$.

b. Describe

i) the transformation of the graph of y_1 to y_2.

ii) the transformation of the graph of y_1 to y_3.

c. Without sketching, describe the transformation of the graph of $y = x^2$ to the graph of $y = 3x^2 + 25$.

d. Describe what happens when a function is multiplied by a constant c, where $c < 0$, $0 < c < 1$, and $c > 1$.

7. a. Sketch the following functions on the same grid:

$y_1 = x^2$, $y_2 = (x - 5)^2$, and $y_3 = -(x + 3)^2$.

b. Describe

i) the transformation of the graph of y_1 to y_2.

ii) the transformation of the graph of y_1 to y_3.

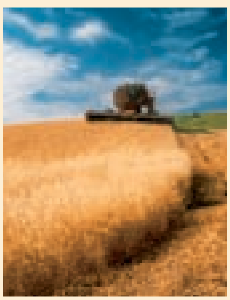
c. Without sketching, describe the transformation of the graph of $y = x^2$ to the graph of $y = (x + 6)^2 - 7$.

d. Describe what happens when a positive or negative constant is added to the variable in a function.

Career Link investigate

CHAPTER 6: DISCOVERING EXPONENTIAL GROWTH PATTERNS

If you have suffered from the bacterial infection streptococcal pharyngitis, better known as strep throat, you have had first-hand experience with exponential growth. Not all applications of exponential growth are so negative, however. Bacteria are used in a multitude of biotechnology applications, including the destruction of hazardous wastes such as PCBs, toxins that otherwise would be very difficult to remove from our natural environment. Other applications with exponential patterns include the mathematics of investment (growth) and the carbon dating of archeological relics (decay). In this chapter, you will investigate patterns in exponential graphs, solve problems following exponential growth and decay patterns, and build and manipulate mathematical models following exponential patterns.



Case Study — Agricultural Entomologist

Faced with changing weather patterns bringing more severe weather events, Canada's agricultural sector is attempting to create disease- and insect-resistant crops that are adaptable to both dry and wet climates. As part of this process, the population dynamics of insects are thoroughly investigated. An entomologist, an insect researcher, is examining the birth and death patterns of an insect that destroys soy crops. The researcher notices that if no deaths were to occur, the population would grow by 50% each day. She has also observed that 10% of the population, including the new births, dies every day. The starting population is 1000.

DISCUSSION QUESTIONS

1. Complete a table showing the population dynamics to day 5 with the following column headings, then draw a rough sketch of the Start-of-Day versus End-of-Day Population.

Day	Population Start of Day	Births	Deaths	Population End of Day
0	—	—	—	1000
	1000			
5				

2. Considering the relationships you have studied to date (linear, quadratic, higher-degree polynomial, periodic, rational, geometric series), what does this pattern most closely resemble? Explain your reasoning.
3. For "steady-state" conditions to exist (no population growth or decline), what percentage of the population would have to die each day? Explain.
4. Identify at least two other relationships in nature or technology that follow exponential growth or decay patterns (that were not mentioned on this page). Explain why these relationships follow exponential patterns.

At the end of this chapter, you will build, modify, and manipulate an algebraic and graphical mathematical population dynamics model for two insects in an ecosystem. ●

Section 6.1 — Laws of Exponents

Your study of calculus will require an ability to manipulate rational and negative exponents. The exponent laws enable us to simplify and evaluate expressions involving exponents. Here is a summary of the exponent laws.

Exponent Laws

$a^m \times a^n = a^{m+n}$	$x^{-n} = \frac{1}{x^n}, x \neq 0$
$\frac{a^m}{a^n} = a^{m-n}, a \neq 0$	$\frac{1}{x^{-n}} = x^n, x \neq 0$
$(a^m)^n = a^{mn}$	$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n, a, b \neq 0$
$(ab)^m = a^m b^m$	$a^{\frac{p}{q}} = (\sqrt[q]{a})^p$ or $\sqrt[q]{a^p}$
$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$	
$x^0 = 1$	Alternatively, $a^{\frac{p}{q}} = \left(a^{\frac{1}{q}}\right)^p$ or $(a^p)^{\frac{1}{q}}$

EXAMPLE 1

Evaluate each of the following:

a. $\frac{5^{-2}}{3^{-3}}$ b. $\frac{2^{-1} + 4^{-1}}{3^{-2}}$ c. $\frac{4^{\frac{3}{2}} - 8^{\frac{1}{3}}}{16^{\frac{1}{4}} \times 25^{\frac{1}{2}}}$

Solution

a. $\frac{5^{-2}}{3^{-3}} = \frac{3^3}{5^2}$

$= \frac{27}{25}$

b. $\frac{2^{-1} + 4^{-1}}{3^{-2}} = \frac{\frac{1}{2} + \frac{1}{4}}{\frac{1}{3^2}}$

$= \frac{\frac{3}{4}}{\frac{1}{9}}$

$= \frac{3}{4} \times \frac{9}{1}$

$= \frac{27}{4}$

c. $\frac{4^{\frac{3}{2}} - 8^{\frac{1}{3}}}{16^{\frac{1}{4}} \times 25^{\frac{1}{2}}} = \frac{8 - 2}{2 \times 5}$

$= \frac{6}{10}$

$= \frac{3}{5}$

EXAMPLE 2

Simplify each of the following, using the laws of exponents.

a. $\sqrt[3]{b^4} \times (b^2)^3$ b. $\frac{a^{\frac{3}{4}} \times a^{\frac{1}{3}}}{a^{\frac{1}{2}}}$

Solution

a. $\sqrt[3]{b^4} \times (b^2)^3 = b^{\frac{4}{3}} \times b^6$

$= b^{\frac{22}{3}}$

b. $\frac{a^{\frac{3}{4}} \times a^{\frac{1}{3}}}{a^{\frac{1}{2}}} = a^{\frac{3}{4} + \frac{1}{3} - \frac{1}{2}}$

$= a^{\frac{9+4-6}{12}}$

$= a^{\frac{7}{12}}$

EXAMPLE 3

Simplify, using the laws of exponents.

a. $x^{\frac{2}{3}} \div x^{-\frac{4}{3}}$

b. $\frac{(x^2y - xy^2)^3}{(xy)^4}$

Solution

a. $x^{\frac{2}{3}} \div x^{-\frac{4}{3}} = x^{\frac{2}{3}-\left(-\frac{4}{3}\right)}$

$= x^2$

b. $\frac{(x^2y - xy^2)^3}{(xy)^4} = \frac{[xy(x - y)]^3}{(xy)^4}$

$= \frac{(xy)^3(x - y)^3}{(xy)^4}$

$= \frac{(x - y)^3}{xy}$

Exercise 6.1

Part A

1. Evaluate each of the following, using the laws of exponents.

a. $(7^3)^2 \div 7^4$
b. $(0.4)^5 \div (0.4)^3$
c. $(\sqrt{3})^5 \times (\sqrt{3})^3$
d. $25^{\frac{3}{2}}$
e. $(-8)^{\frac{2}{3}}$
f. $(-2)^3 \times (-2)^3$
g. $4^{-2} - 8^{-1}$
h. $(a^4 \div a^7) \times a^3$
i. $(0.3)^3 \div (0.3)^5$
j. $(p^2)^3 \div (p^3)^2$
k. $(3^2)^3 \div 3^{-2}$
l. $(3^{-1})^3 \times 3^2$
m. $(-2)^3 \times 2^{-4}$
n. $(2^3)^{-2} \times (2^{-2})^2$
o. $(6^3)^4 \div 12^6$

Knowledge/Understanding

2. Simplify each of the following, using the laws of exponents.

a. $\frac{x^5y^2}{x^3y^4}$
b. $(xy^2)(x^3y^2)$
c. $\frac{(3a^2b)^2}{(ab^2)^3}$
d. $\frac{2^3g^2h^4}{(gh^2)^3}$
e. $(xy^2)^3$
f. $\frac{(b^2)^3c^4}{(bc)^5}$
g. $\frac{5x^3y^{-4}}{2x^{-2}y^2}$
h. $\frac{\pi x^2y}{4xy^3}$
i. $(5x^2y)^{-2}$
j. $(a^2bc^{-1})^3$
k. $(a^2b^{-1})^{-3}$
l. $(ab)^4\left(\frac{a^{-2}}{b^{-2}}\right)^2$

Knowledge/Understanding

3. Simplify each of the following:

a. $(3x^{-2}y^3)^{-1}$
b. $(a^{\frac{1}{4}}b^{-\frac{1}{3}})^{-2}$
c. $\left(\frac{x^{-3}}{x^{-1}}\right)^{-2}$
d. $\frac{(4x^2y^{\frac{1}{3}})^{\frac{1}{2}}}{(8xy^{\frac{1}{4}})^{\frac{1}{3}}}$
e. $\frac{(4a^{-2})(2a^3b^2)}{12a^4b^3}$
f. $\frac{(5x^{-2}y^0)^3}{(25x^2y)^{\frac{1}{2}}}$

Part B

4. Simplify, using the laws of exponents.

a. $(64x^4)^{\frac{1}{2}}$ b. $\sqrt[4]{16^3}$ c. $(27)^{\frac{4}{3}}$

d. $\sqrt{2}a^{\frac{1}{2}} \times \sqrt{32}a^{\frac{3}{4}}$ e. $\sqrt[3]{27p^6}$ f. $\sqrt[5]{32a^{10}}$

g. $a^{3.4} \times a^{2.6}$ h. $\sqrt[3]{5^2} \div \sqrt[4]{5^5}$ i. $(\sqrt[3]{t})^2 \times \sqrt{t^5}$

Application

5. Simplify, using the laws of exponents.

a. $\dfrac{3^{-1} + 3^{-2}}{3^{-3}}$ b. $\dfrac{ab^2c + a^2bc}{abc}$ c. $\dfrac{(p^2q + pq^3)^3}{p^3q^4}$

d. $\dfrac{x^{-2} - x^{-3}}{2x}$ e. $\dfrac{3t - 2t^{-1}}{t^3}$ f. $\dfrac{3p^2 - p^{-3}}{p^4}$

Thinking/Inquiry/ Problem Solving

6. Simplify each of the following:

a. $\dfrac{x^{\frac{3}{2}} - x^{\frac{1}{2}} - x^{-1}}{x^{-\frac{1}{2}}}$ b. $\dfrac{4 - \sqrt{x}}{x^{\frac{3}{2}}}$ c. $\dfrac{x - 9}{x^{\frac{1}{2}} - 3}$ d. $\dfrac{x - 1}{\sqrt{x} - x}$

Communication

7. Using the laws of exponents, explain why $64^{\frac{1}{6}} = 8^{\frac{1}{3}}$.

Section 6.2 — Investigating $f(x) = b^x$

In this section, you will be investigating the exponential function $f(x) = b^x$. Since you will be drawing several curves on each grid, remember to label each curve with its equation.

INVESTIGATION

technology

1. a. Use your graphing calculator or graphing software to draw the graph of $y = 2^x$. In your notebook, sketch $y = 2^x$, showing the scale on the axes.

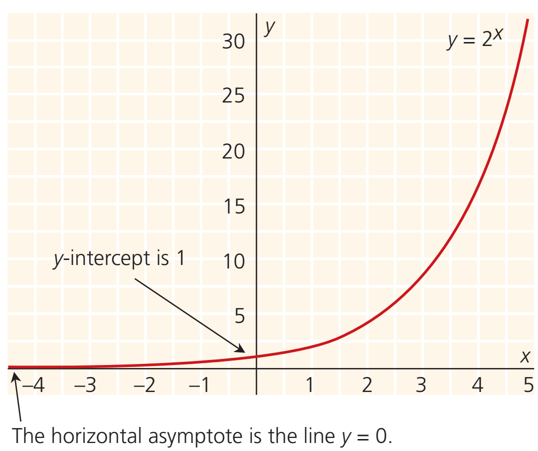
   b. Use the TRACE function to find the value of the y-intercept. Label the y-intercept on your sketch.

   c. Using the TRACE function, move the cursor left. Watch the y-values as the cursor moves. What do you notice about these values? The graph approaches the x-axis for small values of x. The line the graph of a function approaches is called an **asymptote.** Label the horizontal asymptote on your sketch and write its equation.

   d. What is the domain and range of $y = 2^x$?

   e. Does the graph cross the x-axis?

   Your finished sketch should look like this:



Domain is $x \in R$.
Range is $y > 0, y \in R$.

Notice that $y = 2^x$ is an **increasing function.**

2. Using your graphing calculator or graphing software, investigate the function $y = 5^x$ as you did $y = 2^x$. Draw a sketch of $y = 5^x$, showing the scale, and labelling the y-intercept and the horizontal asymptote. State the domain and range of the function.

3. Use your graphing calculator or graphing software to graph $y = 2^x$, $y = 5^x$, and $y = 10^x$. Sketch, labelling the functions carefully.

a. What are the common characteristics of these curves?

b. Predict where the graph of the curve $y = 7^x$ will lie relative to the curves graphed earlier. Check your prediction with your graphing calculator or graphing software.

4. Write a general description of the graph of the exponential curve $y = b^x$, where $b > 1$. In this description include the y-intercept, horizontal asymptote, domain, range, and a sketch.

5. Investigate the function $y = \left(\frac{1}{3}\right)^x$. Draw a sketch of $y = \left(\frac{1}{3}\right)^x$, showing the scale,
and labelling the y-intercept and the horizontal asymptote. State the domain and range of the function. Note that we can express this as $y = 3^{-x}$ rather than $y = \left(\frac{1}{3}\right)^x$.

6. Graph $y = \left(\frac{1}{3}\right)^x$, $y = \left(\frac{1}{5}\right)^x$, and $y = \left(\frac{1}{10}\right)^x$. Sketch, labelling the functions carefully.
   a. What are the common characteristics of these curves?
   b. Predict where the graph of the curve $y = \left(\frac{1}{7}\right)^x$ will be relative to the curves graphed in part **a** of this question. Check your prediction with your graphing calculator or graphing software.

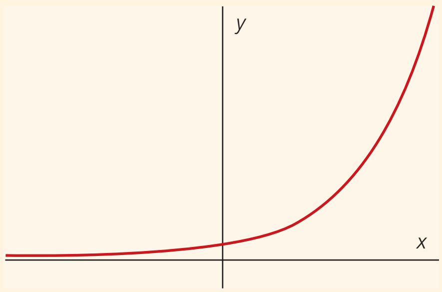
7. Write a general description of the graph of the exponential curve $y = b^x$, where $0 < b < 1$. In this description include the y-intercept, horizontal asymptote, domain, range, and a sketch.

8. Graph $y = 3^x$ and $y = \left(\frac{1}{3}\right)^x$. Sketch, labelling the functions carefully.
   a. What transformation on $y = 3^x$ will give $y = \left(\frac{1}{3}\right)^x$ as its image?
   b. How are the curves alike?
   c. How are they different?

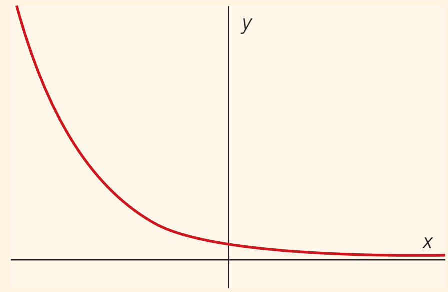
9. Graph $y = 3^x$ and $y = -3^x$.
   a. What transformation on $y = 3^x$ will give $y = -3^x$ as its image?
   b. How are the curves alike?
   c. How are they different?

Properties of the Exponential Function $y = b^x$

- **The base b is positive.**
- **The y-intercept is 1.**
- **The x-axis is a horizontal asymptote.**
- **The domain is the set of real numbers, R.**
- **The range is the set of positive real numbers.**
- **The exponential function is always increasing if $b > 1$.**
- **The exponential function is always decreasing if $0 < b < 1$.**



$y = b^x$, $b > 1$
The function is always increasing.



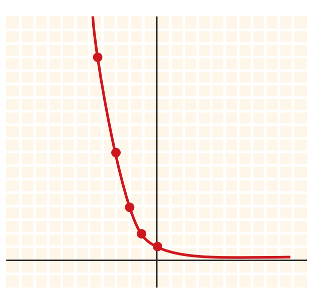
$y = b^x$, $0 < b < 1$
The function is always decreasing.

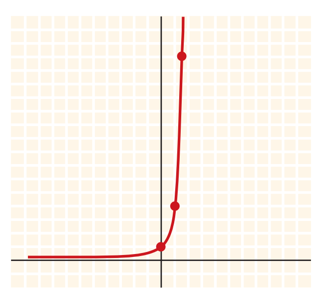
Exercise 6.2

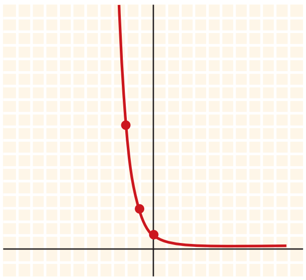
Part A

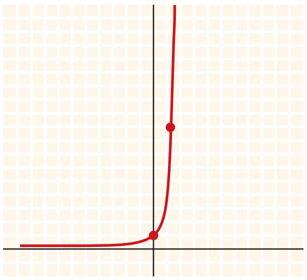
Knowledge/Understanding

1. The graphs below have equations of the form $y = b^x$. Answer the following questions about each of the functions:

a. 

b. 

c. 

d. 

i) What is the value of the y-intercept?

ii) Is the function always increasing or always decreasing?

iii) Is $b > 1$ or is $0 < b < 1$?

iv) What is the value of y when $x = 1$ or $x = -1$?

v) Use the information you gained in parts **i** to **iv** write the equation of the function.

Part B

Application

2. If f is a function defined by $f(x) = b^x$, where $b > 0$, what can be stated about

a. the sign of $f(x)$?

b. the growth behaviour of f?

c. $f(0)$?

Thinking/Inquiry/Problem Solving

3. If f is a function defined by $f(x) = b^x$, where $0 < b < 1$, what can be stated about

a. the sign of $f(x)$?

b. the growth behaviour of f?

c. $f(0)$?

Communication

4. Describe how you determine the equation of a function of the form $y = b^x$, $b > 0$, if you are given its graph. You can use only your knowledge of exponential functions and not your calculator. Include sketches to help you describe the procedure you use.

Section 6.3 — Investigating $f(x) = ab^x + c$

You will be investigating the effect of different transformations on the exponential function $f(x) = b^x$, $b > 0$. Since you will be drawing several curves on each grid, remember to label each curve with its equation. Some coloured pencils will also be helpful. When you are describing transformations, remember to use correct mathematical terms such as **translation, reflection,** or **dilatation.**

INVESTIGATION

technology

1. a. Using your graphing calculator or graphing software, draw the graphs of $f(x) = 2^x$, $g(x) = 2^x + 4$, and $h(x) = 2^x - 3$. Draw a sketch in your notebook and label each function.
   b. Describe the transformation of the graph of $f(x)$ to $g(x)$ and the transformation of $f(x)$ to $h(x)$. What is the effect of adding or subtracting a number from the exponential function?
   c. How is the horizontal asymptote of $g(x)$ and $h(x)$ related to the horizontal asymptote of $f(x)$?

2. a. Using your graphing calculator or graphing software, draw the graphs of $f(x) = 2^x + 4$, $g(x) = 3(2^x) + 4$, and $h(x) = 0.5(2^x) + 4$. Draw a sketch in your notebook and label each function.
   b. Describe the transformation of the graph of $f(x)$ to $g(x)$ and the transformation of $f(x)$ to $h(x)$. What is the effect of multiplying the exponential function by a positive number, where the positive number is greater than one and where the positive number is less than one?
   c. How are the horizontal asymptotes of $g(x)$ and $h(x)$ related to the horizontal asymptote of $f(x)$?

Exercise 6.3

Part A

Knowledge/Understanding

1. For each of the following, state
   i) the equation of the horizontal asymptote,
   ii) whether the function is increasing or decreasing,
   iii) the y-intercept.

a. $y = 3^x - 5$ b. $y = 2^x + 4$ c. $y = 4\left(\frac{1}{3}\right)^x$

d. $y = \left(\frac{1}{2}\right)^x + 2$ e. $y = 2(5^x) - 1$ f. $y = 5\left(\frac{2}{3}\right)^x + 1$

Knowledge/Understanding

2. For the curve $y = 3(4^x) + 5$,

a. determine

i) the horizontal asymptote,

ii) the y-intercept,

iii) whether the curve is increasing or decreasing,

iv) the domain and range.

Application

b. Sketch the curve.

Part B

Knowledge/Understanding

3. For the curve $y = 2\left(\frac{2}{3}\right)^x - 4$,

a. determine

i) the horizontal asymptote,

ii) the y-intercept,

iii) whether the curve is increasing or decreasing,

iv) the domain and range.

Application

b. Sketch the curve.

Communication and Thinking/Inquiry/Problem Solving

4. Describe how you can draw a quick mental picture of the graph for a function of the form $y = ab^x + c$, $b > 0$, by using your knowledge of the effect of changing the parameters a, b, and c. Use specific equations and sketches to help you describe the process.

Section 6.4 — Exponential Growth and Decay

Exponential growth occurs when quantities increase or decrease at a rate proportional to the quantity present. This growth or decay occurs in savings accounts, the size of populations, and the quantity of decay that occurs in radioactive chemicals. All situations of this type can be expressed using the exponential function.

Consider, for example, the population growth of a city. The population is currently 100 000 and is growing at the rate of 5% per year. This is exactly the same situation as the compound interest problems we dealt with in an earlier grade. Each year the population increases by a factor of 1.05, hence the population after one year is 100 000(1.05). After two years, it is (100 000(1.05))1.05 or 100 000$(1.05)^2$. In the same way, a population of 100 000 increasing at 5% for n years grows to 1000 000$(1.05)^n$ people, and the function that expresses the population after x years is $f(x) = 100\,000(1.05)^x$.

EXAMPLE 1

An antique vase was purchased in 2000 for \$8000. If the vase appreciates in value by 6% per year, what is its estimated value in the year 2040, to the nearest thousand dollars?

Solution

The value of the vase is given by $f(x) = 8000(1.06)^x$, where x is the time in years.

$$f(40) = 8000(1.06)^{40}$$
$$\doteq 82\,285.7435$$

In 2040 the vase will be worth approximately \$82 000.

EXAMPLE 2

A very convenient measure of population growth is a doubling period. The population of the world was 6 billion in 1999. This population is growing exponentially and doubles every 35 years.

a. Estimate the world population in 2050, to the nearest half billion.
b. When will the population be 24 billion?

Solution

a. If the population doubles, the base for the function must be 2. However, the time for doubling is 35 years, so the exponent must be of the form $\frac{t}{35}$, where t is the number of years. Then the function representing population after t years is $f(t) = 6\left(2^{\frac{t}{35}}\right)$. For the population in 2050, $t = 51$.

$$f(51) = 6\left(2^{\frac{51}{35}}\right) \doteq 16.5$$

The population in 2050 will be approximately 16.5 billion.

b. To determine t such that $f(t) = 24$, we write

$$6\left(2^{\frac{t}{35}}\right) = 24$$

$$2^{\frac{t}{35}} = 4 = 2^2.$$

Then $\frac{t}{35} = 2$

$$t = 70.$$

The population will be 24 billion in 2069.

EXAMPLE 3

We do not always know the rate of increase. In this case, we construct an exponential function with an unknown base and determine the base from the given information. The population of a town was 24 000 in 1980 and 29 000 in 1990.

a. Determine an expression for the population at the time t years after 1980.
b. Use this expression to estimate the population of the town in 2020.

Solution

a. Let the population in t years be $P(t) = P_0 b^t$, where P_0 is the population at time $t = 0$, with $t = 0$ in 1980. Then $P(0) = P_0 b^0 = P_0 = 24$, in thousands.

Now $P(t) = 24b^t$.

We are given that $P(10) = 29$.

Then $24b^{10} = 29$

$$b^{10} = \frac{29}{24}.$$

Taking roots on both sides,

$$b = \left(\frac{29}{24}\right)^{\frac{1}{10}}$$

$$\doteq 1.019.$$

Now $P(t) = 24(1.019)^t$.

b. In the year 2020, $t = 2020 - 1980 = 40$.

$$P(40) = 24(1.019)^{40}$$

$$\doteq 51.16$$

The population in 2020 will be approximately 51 000.

EXAMPLE 4

Just as population growth and inflation can be described by an exponential growth function, radioactive decay and depreciation can be described by an exponential decay function. A car depreciates by 15% per year. If you buy a car for \$15 000, find the value of the car in three years.

Solution

The car depreciates by 15% per year, so the base for the exponential function is $(1 - 0.15) = 0.85$.

The value of the car when $t = 0$ is 15 000.

The value of the car after t years is $V(t) = 15\,000(0.85)^t$.

Then $V(3) = 15\,000(0.85)^3 = 9211.88$.
After three years, the car is worth approximately \$9200.

Scientists use the term "half-life" when discussing substances that are radioactive, like polonium^{210}. Polonium^{210} has a half-life of 140 d. This means that in 140 d, half of the amount of polonium^{210} will have decayed to some other substance. If we start with 10 g of polonium^{210}, after 140 days we will have 5 g of polonium^{210} left. The other 5 g will have decayed to some other substance.

Energy is released and power is generated when a nucleus decays. Devices have been built to extract this energy in a useful form, such as electricity. When a nucleus decays, energy in the form of heat is also released. Devices that use that heat to generate electricity are useful in specialized applications, such as implantable heart pace-makers, power sources for lunar stations, and isolated automated weather stations.

EXAMPLE 5

An isotope of radium is used by a hospital for cancer radiation. The half-life of this radium is 1620 years. If the hospital initially had 10 mg, how much will they have after 50 years?

Solution

Since the radium has a half-life of 1620 years, the base for the exponential function is $\frac{1}{2}$. After time t in years, the amount of radium $A(t)$ is given by $A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$, where A_0 is the initial amount and h is the half-life.

The hospital initially had 10 mg of radium with a half-life of 1620 years.

Then $A(t) = 10\left(\frac{1}{2}\right)^{\frac{t}{1620}}$.

The amount left after 50 years is $A(50) = 10\left(\frac{1}{2}\right)^{\frac{50}{1620}} = 9.79$ mg.

Exercise 6.4

Part A

1. The population of a city is 810 000. If it is increasing by 4% per year, estimate the population in four years.

2. A painting, purchased for \$10 000 in 1990, increased in value by 8% per year. Find the value of the painting in the year 2000.

3. A river is stocked with 5000 salmon. The population of salmon increases by 7% per year.
   a. Write an expression for the population t years after the salmon were put into the river.

b. What will the population be in

i) 3 years?

ii) 15 years?

c. How many years does it take for the salmon population to double?

Knowledge/Understanding

4. A house was bought six years ago for $175 000. If real-estate values have been increasing at the rate of 4% per year, what is the value of the house now?

5. A used-car dealer sells a five-year-old car for $4200. What was the original value of the car if the depreciation is 15% a year?

Knowledge/Understanding

6. In the early 1990s, the Canadian dollar was declining in value due to inflation at the rate of 8.3% per year. If the situation continued, what would the dollar be worth five years later?

Part B

Application

7. To determine whether a pancreas is functioning normally, a tracer dye is injected. A normally functioning pancreas secretes 4% of the dye each minute. A doctor injects 0.50 g of the dye. Twenty minutes later, 0.35 g remain. If the pancreas were functioning normally, how much dye should remain?

8. If a bacteria population doubles in 5 d,

a. when will it be 16 times as large?

b. when was it $\frac{1}{2}$ of its present population?

c. when was it $\frac{1}{4}$ of its present population?

d. when was it $\frac{1}{32}$ of its present population?

9. Inflation is causing things to cost roughly 2% more per year.

a. A bag of milk costs $3.75 now. Estimate its cost in five years.

b. i) A movie ticket costs $8.50 now. If inflation continues at 2% per year, when will the ticket cost $10.00?

ii) How long ago did the movie ticket cost $4.25?

10. An element is decaying at the rate of 12%/h. Initially we have 100 g.

a. How much remains after 10 h?

b. How much remains after 30 h?

c. When will there be 40 g left?

Application **11.** A research assistant made 160 mg of radioactive sodium (Na^{24}) and found that there was only 20 mg left 45 h later.

a. What is the half-life of Na^{24}?

b. Find a function that models the amount A left after t hours.

c. If the laboratory requires 100 mg of Na^{24} 12 h from now, how much Na^{24} should the research assistant make now? (Ignore the 20 mg she currently has.)

d. How much of the original 20 mg would be left in 12 h?

12. A bacteria colony grows at the rate of 15%/h.

a. In how many hours will the colony double in size?

b. In 10 h the bacteria population grows to 1.3×10^3. How many bacteria were there initially?

Thinking/Inquiry/Problem Solving **13.** People who work frequently in a radiation environment, such as X-ray technicians, dentists, radiologists, or nuclear reactor operations staff, are limited to a 50 mSv (milli-sievert) whole-body radiation dose in any one given year. They wear badges that measure the radiation to which they have been exposed. We all receive radiation from many sources, both naturally occuring and artificial, all the time. The North American average from all sources is approximately 2 mSv/year. We can find these levels increasing if we mountain climb or smoke.

a. Calculate your own radiation exposure in a year, using the information below.

Radiation Source	Amount of Radiation (in milli-sieverts per year)
Cosmic and terrestrial	0.8
Atmospheric fallout	0.05
Internal body	0.3
Living above sea level	0.02 for every 100 m above sea level
Chest or dental x-ray	0.1 for every x-ray
TV watching	0.003 for every 3 h/d
Housing	0.06 if house contains masonry
Flying	0.001 for every hour spent flying
Cigarette smoking (heavy smoker)	60

Communication **b.** How can you reduce your exposure to radiation?

Communication **c.** Is there anywhere that is radiation-free?

Application

14. The population of a city was estimated to be 125 000 in 1930 and 500 000 in 1998.

a. Estimate the population of the city in 2020.

b. If the population continues to grow at the same rate, when will the population reach 1 million?

Part C

Thinking/Inquiry/Problem Solving

15. On the day his son is born, an excited father wants to give his new son a season's ticket to watch the father's favourite sports team. A season's ticket costs \$900. The father realizes there is no point in buying tickets for a baby only a few hours old, so he decides to put the money aside until the boy is six years old. If inflation is assumed to be 3% per year, how much money should the father put aside so that he can purchase the season's ticket in six years?

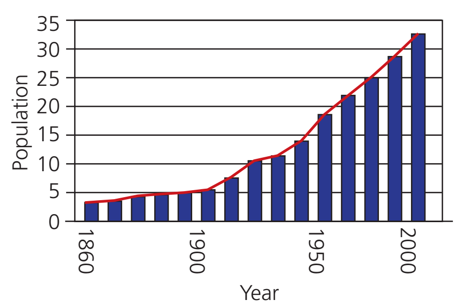
Thinking/Inquiry/Problem Solving

16. Two different strains of cold virus were isolated and put in cultures to grow. Virus A triples every 8 h while virus B doubles every 4.8 h. If each culture has 1000 viruses to start, which has more after 24 h?

Thinking/Inquiry/Problem Solving

17. With exponential growth, a population will continue to double in equal intervals of time. However, in a finite world, one population influences the growth of another. A garden pond may have water lilies covering part of the surface. Growing conditions are ideal and the number of water lilies doubles, and doubles again. Then the gardener, realizing that the water lilies will soon cover the whole pond, introduces a chemical to kill off many of the plants. The exponential growth pattern is disrupted. But this disruption in the exponential growth pattern would occur without the intervention of the gardener. As the water lilies covered more and more of the pond surface, the plants would compete for food and light. The over-crowding of the plants would reduce the rate of expansion. This levelling off occurs in every example of natural growth.

Below is a graph of the Canadian population between the years 1860 and 2000.



Communication

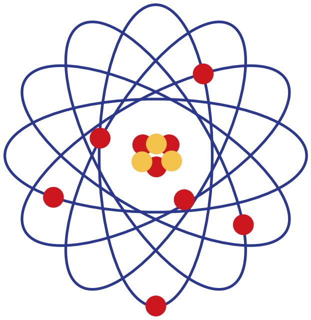
a. Why do you think the population of Canada is not growing as fast now as it was earlier in the twentieth century?

Communication

b. Discuss the factors that affect the population of a country.

Section 6.5 — Modelling Data Using the Exponential Function

At the turn of the last century, scientists discovered that certain naturally occurring substances emitted invisible and penetrating radiation. In such substances, the nucleus spontaneously undergoes a rearrangement that may include the ejection of some nuclear particles and possibly gamma rays. Strontium^{90} is such a substance. Below is a set of measurements for the amount of strontium^{90} remaining after several time intervals have passed. You will be using your graphing calculator to model this data.



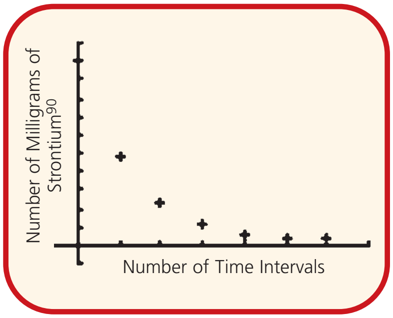
INVESTIGATION

Initially (at time zero) we have 500 mg of strontium^{90}. Here are the results of measuring the amount of strontium^{90} remaining after 6 time intervals have passed.

Number of Time Intervals	Amount of Strontium^{90} (in milligrams)
0	500
1	241
2	112
3	61
4	30
5	18
6	13

technology

a. Use your graphing calculator to draw a scatter plot of the data. Adjust your viewing window so that the graph looks similar to this. Sketch the graph in your notebook. Remember to include the scale and the labels on the axes.



Technical Help

To enter the data, press STAT. Under EDIT, select **1:Edit**. Enter the values in L1 and L2.

To draw the scatter plot, turn off all functions in Y=. Press STATPLOT Y= to obtain the STAT PLOT function. Select **1:Plot 1** and press ENTER. Turn plot 1 on, select scatter plot, and L1 and L2 for Xlist and Ylist. Press WINDOW to adjust the viewing window. Press GRAPH to see the scatter plot.

b. We now want to model this data with a function. If you pictured a curve passing through our points, it would be shaped somewhat like the exponential curve. We will find the curve of best fit determined by your calculator. Use the exponential regression calculation. To three decimal places, the equation is $y = 431.856(0.536)^x$.

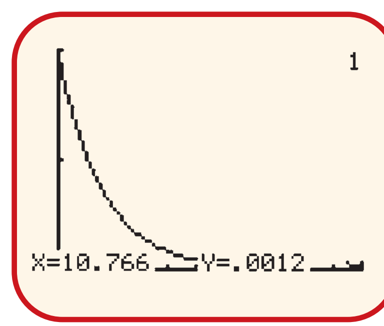
Technical Help

To use the exponential regression calculation, press STAT, then under CALC, select **0:ExpReg**. Press ENTER twice. The coefficient and the base of the exponential regression equation are displayed.

c. Using the equation $y = 431.856(0.536)^x$, which we found as a model of the data, estimate the amount of strontium^{90} remaining after 15 time intervals.

d. Using the equation $y = 431.856(0.536)^x$, estimate when there will be 0.5 mg of strontium^{90} remaining.

Use the graphing calculator to draw the graph of $y = 0.536^x$. We can use the TRACE feature of the calculator to approximate the solution to $0.536^x = 0.0012$.



The solution to $0.536^x = 0.0012$ is approximately $x \doteq 10.766$. There will be 0.5 mg of strontium^{90} remaining after approximately 10.8 time intervals. The accuracy of your answer can be improved by using the ZOOM feature of the calculator.

Alternately, you can guess-and-check using the power key on the calculator to find the solution to $0.536^x = 0.0012$.

$$0.536^{10} = 0.0020$$
$$0.536^{11} = 0.0010$$

The value of x must be between 10 and 11. Continue to experiment with values to improve the accuracy of your approximation.

Strontium^{90} has a half-life of 26.8 years. In one half-life, half of this radioactive material will have decayed to some other substance. If we started with 10 g of strontium^{90}, after 26.8 years there should be 5 g left. In our calculations, each time interval represented 26.8 years.

As the term "half-life" indicates, our expression should be of the form $y = a\left(\frac{1}{2}\right)^x$. Since we started with 500 g of strontium^{90}, the expression should be $y = 500\left(\frac{1}{2}\right)^x$.

But things are never quite that precise when data is collected experimentally. Using experimental data allows you to get close to the theoretical equation. The more data points you have, and the more accurate the points are, the better the actual situation will fit the mathematical model.

It is unusual to obtain a good function approximation using $y = b^x$. Most data sets require an exponent of a more complicated form than x, such as $3x^2 - 2x$. Such situations are beyond the scope of this text.

Exercise 6.5

Part B

Knowledge/Understanding

1. A population of bacteria, initially 1000, is growing. The size of the population is measured every hour. The results are shown in the table below.

Number of Time Intervals	Bacteria Population
0	1000
1	1135
2	1307
3	1490
4	1696
5	1957
6	2228

technology

a. Use your graphing calculator to draw a scatter plot of the data. Sketch the scatter plot in your notebook. Include the labels and scales on the axes.

b. Using the exponential regression calculation in your calculator, determine the equation of the curve of best fit. Record this equation accurate to three decimal places.

c. Using the equation of the curve of best fit, estimate the bacteria population in 10 h.

d. Predict when there would be 10 000 bacteria. *Hint:* How many time intervals will this take?

2. Below is a table showing the population of the world. This information came from the United Nations Web site, www.un.org/popin. We will count in intervals of 50 years from 1750.

Year	Time Interval	Population (in billions)
1750	0	0.79
1800	1	0.98
1850	2	1.26
1900	3	1.65
1950	4	2.52
2000	5	6.06

Knowledge/Understanding

a. With your graphing calculator, draw a scatter plot of the data. Using the exponential regression calculation in your calculator, determine the equation of the curve of best fit accurate to three decimal places.

b. Using your mathematical model, estimate the world population in 2050.

Thinking/Inquiry/Problem Solving

c. Predict when there would be 7 billion people on the earth.

3. The table below shows the carbon dioxide concentration in the atmosphere in parts per million. We will count in intervals of 20 years from 1860.

Year	Time Interval	Carbon-Dioxide Concentration (in parts per million)
1860	0	294
1880	1	296
1900	2	300
1920	3	307
1940	4	308
1960	5	319
1980	6	340
2000	7	377

a. With your graphing calculator, draw a scatter plot of the data. Using the exponential regression calculation in your calculator, determine the equation of the curve of best fit accurate to three decimal places.

b. Using your mathematical model, estimate the carbon-dioxide concentration in 1930 and in 1990.

c. If the trend continues, predict when the concentration will be 390 parts per million.

4. The table below shows the amount of stored nuclear waste in million curies. We will count in intervals of five years from 1970.

Year	Time Intervals	Stored Nuclear Waste (in million curies)
1970	0	5
1975	1	30
1980	2	100
1985	3	210
1990	4	360
1995	5	660

a. With your graphing calculator, draw a scatter plot of the data. Using the exponential regression calculation in your calculator, determine the equation of the curve of best fit accurate to three decimal places.

b. Using your mathematical model, estimate the amount of nuclear waste stored in 1983.

c. If this trend continues, predict when the amount of nuclear waste stored will be 800 million curies.

5. Statistics Canada is a government agency that collects and analyzes data about many aspects of life in Canada. On their Web site, www.statcan.ca, you can find the population of the provinces for the last few years. Select one of the provinces. Copy and complete the chart below using the data from the Statistics Canada Web site. Count the time intervals of years from the first entry in your chart.

Year	Time Intervals	Population

a. With your graphing calculator, draw a scatter plot of the data. Using the exponential regression calculation in your calculator, determine the equation of the curve of best fit accurate to three decimal places.

b. Using your mathematical model, estimate the population of that province in 1900.

c. Predict when the population of that province will be 10% greater than its current level.

Communication

6. Given a set of data, describe how you can predict the algebraic form of a mathematical model to fit the data. Use sketches to illustrate your answer.

Part C

Thinking/Inquiry/
Problem Solving

7. You have likely observed that the models obtained by using your calculator provide a rather poor fit for the data. This is because your calculator uses only one exponential function, namely $f(x) = Ab^x$. By changing the exponent to $px^2 + qx$, a much better fit of the data can be obtained. You may be interested in using more sophisticated exponents with the data given in the problems of this section to see whether you can determine functions that better describe the situation. This will require a computer with algebraic capability.

Key Concepts Review

In Chapter 6, you learned how to identify key properties of exponential functions, and to determine intercepts and positions of the asymptotes to a graph. You should now know how to describe graphical implications of changes in parameters, as well as how to describe the significance of exponential growth or decay. You should not only be able to pose and solve problems related to models of exponential functions, but also to predict future behaviour by extrapolating from a mathematical model. Here is a brief summary of the concepts covered in this chapter.

Exponent Laws

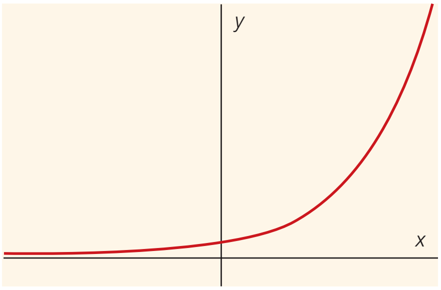
- $a^m \times a^n = a^{m+n}$
- $\frac{a^m}{a^n} = a^{m-n}, a \neq 0$
- $(a^m)^n = a^{mn}$
- $(ab)^m = a^m b^m$
- $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$
- $x^0 = 1$
- $x^{-n} = \frac{1}{x^n}, x \neq 0$
- $\frac{1}{x^{-n}} = x^n, x \neq 0$
- $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n, a, b \neq 0$
- $a^{\frac{p}{q}} = (\sqrt[q]{a})^p$ or $\sqrt[q]{a^p}$

Alternatively,

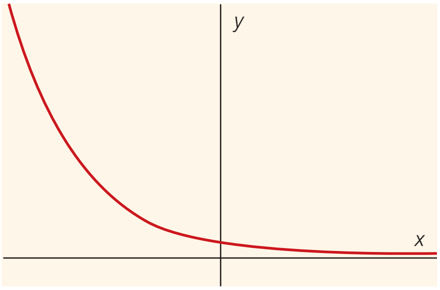
- $a^{\frac{p}{q}} = \left(a^{\frac{1}{q}}\right)^p$ or $(a^p)^{\frac{1}{q}}$

Properties of the Exponential Function $y = b^x$

- The base b is positive.
- The y-intercept is 1.
- The x-axis is a horizontal asymptote.
- The domain is the set of real numbers, R.
- The range is the set of positive real numbers.
- The exponential function is always increasing if $b > 1$.
- The exponential function is always decreasing if $0 < b < 1$.



$y = b^x, b > 1$
The function is always increasing.



$y = b^x, 0 < b < 1$
The function is always decreasing.

Exponential Growth and Decay

Exponential growth occurs when quantities increase or decrease at a rate proportional to the quantity. This growth or decay occurs in savings accounts, the size of populations, and the quantity of decay that occurs in radioactive chemicals. All situations of this type can be expressed using the exponential function. Data on savings accounts, population growth, or radioactive decay can be modelled using an exponential function.

Career Link wrap-up investigate and apply

CHAPTER 6: DISCOVERING EXPONENTIAL GROWTH PATTERNS

An entomologist is studying the predator–prey relationship for two colonies of insects. She is investigating the possibility of introducing the predator insect in corn farming areas to control the population of the prey insect, which is a nuisance to corn crops.

a. In the experiment, the predator insect doubled in population every three days and the prey insect quadrupled in nine days. The initial population of the predator insect was 500 and the initial population of the prey insect was 1000. At what time were the two populations equal?

b. Once the populations are equal, the prey population depends on births (which would still cause a doubling every three days if there were no predator) and deaths caused by the predator, which amount to 5.0% of the population per day. The researcher needs to develop a single algebraic model to predict the population of the prey insect as a function of births and deaths:

$P = P_o$ (Exponential Growth Due to Births)(Exponential Decay Due to Deaths)

Expressed in doubling times format:

$$P(t) = P_o 2^{kt},$$

where $P(t)$ is the population at time t, in days, P_o is the initial population, and k is growth rate constant.

Hints: 1. Set $t = 0$ for the time when the populations were equal as determined in part **a.**

2. Convert growth expression to base of 2 with exponent laws.

3. Convert death expression to base of 2 using the graph of $y = 2^x$ and finding the value of x for $2^x = (1 - \textit{Decay Rate})$.

c. The experiment will be considered a success if the population of the prey insect is less than 6800 five days after the time the two populations were equal, and if the doubling time of the prey is now more than seven days. Use the expression developed in part **b** and the graphing calculator to judge the experiment's success.

Review Exercise

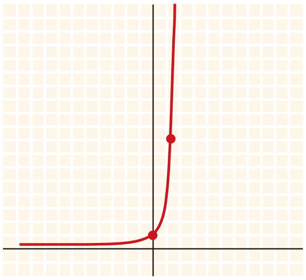
1. Evaluate each of the following:
   a. $(3^{-2} + 2^{-3})^{-1}$
   b. $\dfrac{3^{-3}}{3^{-1} - 3^{-2}}$
   c. $\dfrac{3^{-8}}{3^{-6} \times 3^{-5}}$
   d. $(5^3 - 5^2)(2^3 - 2^2)$

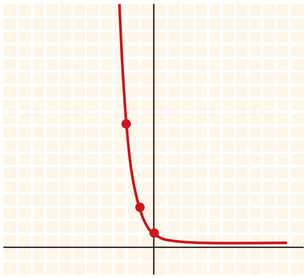
2. Evaluate each of the following:
   a. $32^{-\frac{3}{5}}$
   b. $\left(\dfrac{54}{250}\right)^{\frac{2}{3}}$
   c. $\sqrt[4]{\dfrac{1}{16}}$
   d. $\left(\dfrac{2}{3}\right)^{-3} - 4^{-1}$

3. Simplify each of the following:
   a. $a^{p2+q2} \div a^{p2-q2}$
   b. $\sqrt[3]{\dfrac{x^{\frac{1}{3}}\sqrt{x}}{\sqrt[3]{x^2}}}$
   c. $\dfrac{(x^{a-b})^{a+b}}{x^{a^2}}$
   d. $(16^{p+q})(8^{p-q})$

4. Fully factor each of the following:
   a. $1 + 8x^{-1} + 15x^{-2}$
   b. $x^{\frac{1}{2}} - x^{\frac{5}{2}}$
   c. $x^{-1} + x^{-2} - 12x^{-3}$
   d. $x^{\frac{3}{2}} - 25x^{-\frac{1}{2}}$

5. Following are the graphs of some exponential functions with equations of the form $y = b^x$, $b > 0$. Using what you have learned about exponential functions, and without the aid of your calculator, write the equation of each function. Each line on the graph represents one unit.

   a. 

   b. 

6. For the curve $y = 2(5^x) - 6$,
   a. determine
      i) the horizontal asymptote,
      ii) the y-intercept,
      iii) whether the curve is always increasing or always decreasing,
      iv) the domain and range.

b. Sketch the graph of the function.

7. For the curve $y = 5\left(\frac{1}{2}\right)^x + 3$,

   a. determine

      i) the horizontal asymptote,

      ii) the y-intercept,

      iii) whether the curve is always increasing or always decreasing,

      iv) the domain and range.

   b. Sketch the graph of the function.

8. For a biology experiment, there are 50 cells present. After 2 h there are 1600 bacteria. How many bacteria would there be in 6 h?

9. A laboratory has 40 mg of iodine 131. After 24 d there are only 5 mg remaining. What is the half-life of iodine 131?

10. The chart below shows the population of Canada. This information is from the Statistics Canada Web site at www.statcan.ca. The time intervals are in years, from 1994.

Year	Time Intervals	Population (in millions)
1994	0	29 036.0
1995	1	29 353.9
1996	2	29 671.9
1997	3	30 011.0
1998	4	30 301.2

technology

   a. Using your graphing calculator, draw a scatter plot of this data. Use the exponential regression function on your calculator to find the curve of best fit for the data.

   b. Using your mathematical model, estimate the population of Canada in 2010.

   c. Predict when there will be 35 million people in Canada.

11. The following information is from the United Nations Web site, www.un.org/popin.

   a. The following chart gives the population of Europe.

Year	Population (in millions)
1750	163
1800	203
1850	276
1900	408
1950	547
1998	729

i) Find the average rate of change in population between 1750 and 1800.

ii) Find the average rate of change in population between 1950 and 1998.

iii) Compare the answers to parts **i** and **ii**.

b. The chart below gives the population of North America.

Year	Population (in millions)
1750	2
1800	7
1850	26
1900	82
1950	172
1998	305

i) Find the average rate of change in population between 1800 and 1850.

ii) Find the average rate of change in population between 1950 and 1998.

iii) Compare the answers to parts **i** and **ii**.

c. Between 1800 and 1850, the average rate of change in population in Europe was greater than the average rate of change in population in North America. However the change in the size of the population in North America was far more dramatic. Explain.

Chapter 6 Test

Achievement Category	Questions
Knowledge/Understanding	All questions
Thinking/Inquiry/Problem Solving	10
Communication	4, 9c, 10b
Application	3, 4, 5b, 6, 7, 8, 9b

1. Evaluate each of the following:
   a. $\left(4^{\frac{1}{2}}\right)^3$ b. $(5^{\frac{1}{3}} \div 5^{\frac{1}{6}})^{12}$ c. $4^{-1} + 2^{-3} - 5^0$
   d. $(\sqrt{2})^3 \times (\sqrt{2})^5$ e. $\dfrac{2^{-1} + 2^{-2}}{2^{-3}}$ f. $(-5)^{-3} \times (5)^2$
2. Simplify each of the following:
   a. $\dfrac{a^4 \bullet a^{-3}}{a^{-2}}$ b. $(3x^2y)^2$ c. $(x^4y^{-2})^2 \bullet (x^2y^3)^{-1}$
   d. $(x^{a+b})(x^{a-b})$ e. $\dfrac{x^{p^2-q^2}}{x^{p+q}}$ f. $\sqrt{\sqrt{x} \bullet \sqrt[3]{x}}$
3. Write $\dfrac{x - 16}{x^{\frac{1}{2}} - 4}$ as a polynomial expression.
4. If f is a function defined by $f(x) = b^x$, describe the growth behaviour of $f(x)$. Consider cases if required.
5. For the curve $y = 2\left(\frac{1}{3}\right)^x - 5$,
   a. determine
      i) the horizontal asymptote,
      ii) the y-intercepts,
      iii) whether the curve is increasing or decreasing,
      iv) the domain and range.
   b. Sketch the curve.
6. An antique dresser was purchased for \$3500 in 1985. The dresser increases in value by 7% per year. Find the value of the dresser in 2002.
7. The population of a fishing village is decreasing by 8% per year. In 1998 there were 4500 people living in the village. Estimate the population in 2004.

8. At the end of 14 min, $\frac{1}{16}$ of a sample of polonium remains. Determine the half-life of polonium.

technology

9. Below is a table showing the population of the world. This information came from the United Nations Web site, www.un.org/popin. The data is in intervals of 50 years from 1750, with 50 years considered as one time interval.

Year	Time Intervals	Population (in billions)
1750	0	0.79
1800	1	0.98
1850	2	1.26
1900	3	1.65
1950	4	2.52
2000	5	6.06

a. Use your graphing calculator to draw a scatter plot of the data. Using the exponential regression calculation in your calculator, determine the equation of the curve of best fit accurate to three decimal places.

b. Using your mathematical model, estimate the world population in 2300.

c. If the habitable surface area of the earth is about 20 million hectares, what will be the population density in the year 2300? (1 ha = 10 000 m^2)

d. Do you think the exponential model determined by the graphing calculator is valid over an extended period? Explain your answer.

10. This graph has an equation of the form $f(x) = b^x + c$.

a. Determine values for b and c.

b. Explain how you arrived at this answer.